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A Discrete Spectrum of Solutions of the Wave Equation with Strong Cubic Nonlinearity

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Abstract

The nonlinear wave equation, $\phi_{tt} - \Delta\phi + \phi^3 = 0$, has many solutions that are periodic in time and localized in space, all with infinte energies. The search for spherically symmetric solutions that are well represented by the simple approximation, $\phi(r, t) \equiv A(r) \sin \omega t$, leads to a discrete spectrum of solutions $\{\phi_N(r, t; \omega)\}$. The solutions are nonlinear wave-packets, and they can be regarded as particles. The asymptotic theory ($\omega \to \infty$) of the motion of the guiding center of the Nth wavepacket, in the presence of a specified potential, is characterized by an infinite mechanical mass and an infinte interaction mass, and they are compatible. The rest mass in the classical relativistic mechanics of guiding centers is $m_0 c^2 = \hbar_N \omega$; i.e. the spectrum $\{\phi_N\}$ determines a spectrum of Planck's constants.

1. *Introduction–properties of periodic solutions*

The nonlinear wave equation,

$$\phi_{tt} - \Delta\phi + \phi^3 = 0 \tag{1.1}$$

is often referred to as the *classical* $(\phi^4)_4$–*theory*–classical because the Cauchy problem, $\phi(\mathbf{x}, 0)$ and $\phi_t(\mathbf{x}, 0)$ specified, is well posed in it (Strauss, 1968), and $(\phi^4)_4$ in accord with its variational formulations in terms of

$$W[\phi] \equiv \int \{\tfrac{1}{2}(\phi_t^2 - |\nabla\phi|^2) - \tfrac{1}{4}\phi^4\}\, d^3x\, dt \tag{1.2}$$

At first we shall deal with general properties of spherically symmetric, periodic solutions of the Euler equation (1.1), then, in Section 5 a variational formulation will be introduced to provide a further restriction of the class of solutions.

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The spherical solutions of (1.1) satisfy

$$\phi_{rr} + \frac{2}{r}\phi_r - \phi_{tt} = \phi^3 \tag{1.3}$$

and the conservation laws (cf. Noether's Th.),

$$\begin{gathered} \rho_t + j_r + \frac{2}{r} j = 0 \\ j_t + \sigma_r + \frac{2}{r}\phi_r^2 = 0 \end{gathered} \tag{1.4}$$

where

$$\begin{aligned} \rho &\equiv \tfrac{1}{2}(\phi_t^2 + \phi_r^2) + \tfrac{1}{4}\phi^4 \\ \sigma &\equiv \tfrac{1}{2}(\phi_t^2 + \phi_r^2) - \tfrac{1}{4}\phi^4 \\ j &\equiv -\phi_t\phi_r \end{aligned} \tag{1.5}$$

In accord with (1.4) we shall define the total energy of a solution as

$$E \equiv \lim_{R \to \infty} E_R, \qquad E_R \equiv 4\pi \int_0^R \rho r^2 \, dr \tag{1.6}$$

The reason for the limit in (1.6) will become evident presently; at this point let it be noted that

$$\frac{d}{dt} E_{R(t)} = 0 \leftrightarrow \rho(R(t), t)\frac{dR}{dt} = j(R(t), t) \tag{1.7}$$

In particular, for a periodic solution, where $\rho(r, t)$ and $j(r, t)$ have nonzero and zero mean values, respectively, $R(t)$ is periodic and the mean value of the energy within a sphere of radius R is constant.

The spherically symmetric periodicity condition is

$$\phi(r, t) = \bar{\phi}(r) + \tilde{\phi}(r, \omega(r)t + \delta(r)) \tag{1.8}$$

where $\bar{\phi}$ is the mean value and $\tilde{\phi}(r, \theta)$ is a 2π-periodic, zero-mean function of θ. The substitution of (1.8) in (1.3) implies significant restrictions: when the result is multiplied by $\tilde{\phi}_\theta$ and integrated over a period, it follows that

$$\left(r^2 [t\omega'(r) + \delta'(r)] \int_0^{2\pi} \tilde{\phi}_\theta^2 \, d\theta\right)_r = 0 \tag{1.9}$$

and, in turn, that the only periodic solutions that are regular at $r = 0$ are those for which $\omega'(r) = \delta'(r) = 0$; i.e. the frequency and phase of the oscillation are *constants,* to which we shall assign the values *one* and *zero*, for convenience.

The second restriction follows from the equation,

$$\bar{\phi}_{rr} + \frac{2}{r}\bar{\phi}_r = \left(\bar{\phi}^2 + \frac{3}{2\pi}\int_0^{2\pi} \tilde{\phi}_t{}^2\, dt\right)\bar{\phi} \tag{1.10}$$

which has no solutions, save $\bar{\phi} = 0$, that remain bounded for $0 \leqslant r < \infty$, regardless of the behavior of $\tilde{\phi}$. Thus the condition that $\bar{\phi}$ shall be zero is necessary for the existence of regular solutions.

All told, the conditions on the solutions to be discussed here are:

I. Spherical; $\phi_{rr} + \frac{2}{r}\phi_r - \phi_{tt} = \phi^3$,

II. Periodic; $\phi(r, t + 2\pi) - \phi(r, t) = \phi(r, 0) = \bar{\phi}(r) = 0$,

III. Regular $(r = 0)$; $\phi_r(0, t) = 0$,

IV. Regular $(r \to \infty)$; $\phi(r, t) \to 0$ as $r \to \infty$.

To each solution of (I-IV) there corresponds an eight-parameter family of solutions of (1.1), generated by the Poincaré group (with seven parameters in place of the usual ten because of the spherical symmetry) and the dimensional group,

$$\phi(r, t) \to \bar{\phi}(r, t) = \omega\phi(\omega r, \omega t) \tag{1.11}$$

which generates solutions with an arbitrary frequency. A typical example is

$$\begin{aligned} \phi(r, t) &\to \bar{\phi}(\mathbf{x}, t) = \omega\phi(\bar{r}, \bar{t}), \\ \bar{r} &= \omega\left(x^2 + y^2 + \frac{(z - \beta t)^2}{1 - \beta^2}\right)^{1/2} \\ \bar{t} &= \omega\frac{t - \beta z}{\sqrt{1 - \beta^2}} \end{aligned} \tag{1.12}$$

with two of the eight parameters—the remaining six are the displacements, $t \to t - t_0$ and $\mathbf{x} \to \mathbf{x} - \mathbf{x}_0$, and the polar and azimuthal angles of the z-axis.

Given a solution of (I-IV), the corresponding solution (1.12) is a wave-packet, with a spheroidal envelope that moves with the constant velocity β (the carrier wave has the frequency $\omega/\sqrt{(1 - \beta^2)}$ and it propagates through the envelope with the phase velocity $1/\beta$)—in effect, a free particle of the classical $(\phi^4)_4$-theory.

2. *Existence—further properties*

It has been shown elsewhere (Bisshopp, 1971a) that there are uncountably many solutions of I-IV. In brief the existence proof goes as follows:

Equation (1.3) is hyperbolic, and the Cauchy problem, $\phi_r(0, t) = 0$ and

$\phi(0, t) = f(t)$, is well posed in a neighborhood of $r = 0$. We seek conditions on the initial data $f(t)$ such that the solution of I-III is bounded on the strip, $0 \leqslant t \leqslant 2\pi$ and $0 \leqslant r < \infty$, and satisfies IV. The method is by construction of Lyapunov functionals, the first of which is the average of the second conservation law over a period. Let $\langle \cdot \rangle$ denote the average over a period, and let

$$\begin{aligned} S[\phi] &\equiv \langle \sigma \rangle = \tfrac{1}{2}\langle {\phi_r}^2 \rangle + V[\phi] \\ V[\phi] &\equiv \tfrac{1}{2}\langle {\phi_t}^2 \rangle - \tfrac{1}{4}\langle \phi^4 \rangle \end{aligned} \tag{2.1}$$

When ϕ is a solution of (1.3) the rate of change of S with r is

$$\frac{dS}{dr} = -\frac{2}{r}\langle {\phi_r}^2 \rangle \leqslant 0 \tag{2.2}$$

The functional $V[\phi]$ depends parametrically on r; its properties, such as the functions for which it is stationary, are independent of r.

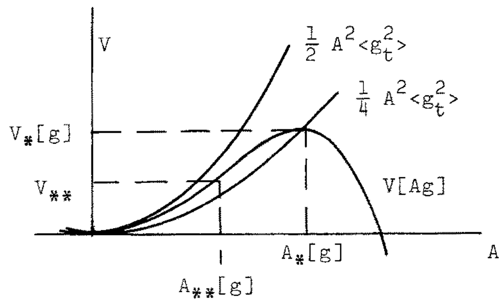

Figure 1 – $V[Ag]$ for fixed $g(r, t)$.

In terms of the representation,

$$\phi(r, t) = A(r)g(r, t), \qquad \langle g \rangle = 0, \qquad \langle g^2 \rangle = 1 \qquad 4 \tag{2.3}$$

$V[Ag]$ varies with A as depicted in Fig. 1.

The value marked V_{**} in Fig. 1 is the minimum over all choices of g of $V_*[g]$; it is the height of the lowest col ($\delta V = 0$), and it is positive ($\approx \frac{1}{6}$ for $g \approx \sqrt{2} \sin t$).

To find bounded solutions we consider the initial data,

$$\phi_r(0, t) = 0, \qquad \phi(0, t) = \mathrm{A}g(t), \qquad 0 < A < A_{**}[g] \leqslant A_*[g] \tag{2.4}$$

Then

$$V_{**} > V(0) = S(0) \geqslant S(r) \geqslant V(r) \geqslant 0 \tag{2.5}$$

i.e. the solution is trapped† in the basin of $\phi = 0$ where

$$\begin{aligned} \langle {\phi_r}^2 \rangle &\leqslant 2S(r) < 2V_{**} \\ \langle {\phi_t}^2 \rangle &\leqslant 4V(r) \leqslant 4S(r) \end{aligned} \tag{2.6}$$

† The domain of trapped solutions is considerably larger than that defined by (2.4).

A further, less obvious property of the trapped solutions is the existence of a positive number ϵ (in the range $(0, \frac{1}{2})$ and independent of r) such that

$$\epsilon\langle\phi_t^2\rangle < \langle\phi_t^2\rangle - \langle\phi^4\rangle, \qquad 0 \leqslant r \leqslant \infty \tag{2.7}$$

The result follows from the observation that $A\ \partial V[Ag]/\partial A \geqslant 0$ when $0 < |A| < A_*[g]$ and $A\ \partial V/\partial A \geqslant \frac{1}{2}A^2\langle g_t^2\rangle$ in a neighborhood of $A = 0$.

To see that the trapped solutions satisfy IV we introduce

$$\bar{S}[\phi] = S[\phi] + \frac{\beta}{r}\langle\phi\phi_r\rangle + \frac{3}{2}\frac{\beta}{r^2}\langle\phi^2\rangle, \qquad \beta > 0 \tag{2.8}$$

such that

$$\frac{d\bar{S}}{dr} = -\frac{2-\beta}{r}\langle\phi_r^2\rangle - \frac{\beta}{r}(\langle\phi_t^2\rangle - \langle\phi^4\rangle) - \frac{3\beta}{r^3}\langle\phi^2\rangle$$

From the Poincaré and Schwartz inequalities (recall $\langle\phi\rangle = 0$) and (2.5),

$$\begin{gathered}\langle\phi^2\rangle \leqslant \langle\phi_t^2\rangle \leqslant 4S(r) \\ \langle\phi\phi_r\rangle^2 \leqslant \langle\phi^2\rangle\langle\phi_r^2\rangle \leqslant 8S(r)^2\end{gathered} \tag{2.9}$$

or

$$\bar{S}(r) = S(r)\left(1 + \frac{\lambda(r)}{r} + \frac{\mu(r)}{r^2}\right) \tag{2.10}$$

$$|\lambda(r)| \leqslant 2\sqrt{2}\beta, \qquad 0 \leqslant \mu(r) \leqslant 6\beta$$

Given $\epsilon > 0$ and (2.6), the choice $2 - \beta = \epsilon\beta$ implies

$$\frac{1}{\bar{S}}\frac{d\bar{S}}{dr} \leqslant \frac{-4\epsilon}{1+\epsilon}\left(r + \lambda(r) + \frac{\mu(r)}{r}\right)^{-1}, \qquad r > \frac{4\sqrt{2}\epsilon}{1+\epsilon} \tag{2.11}$$

from which it follows that S, $\langle\phi_t^2\rangle$, $\langle\phi_r^2\rangle$ and ϕ approach zero as $r \to \infty$.

Once again from the Poincaré and Schwartz inequalities and (2.5),

$$\begin{gathered}\langle\phi^4\phi_t^2\rangle \leqslant \langle\phi^2\rangle^2\langle\phi_t^2\rangle \leqslant 64S(r)^3 \\ \langle\phi^4\phi_r^2\rangle \leqslant 32S(r)^3\end{gathered} \tag{2.12}$$

Thus the Fourier series,

$$\begin{gathered}\phi = \Sigma\, a_n(r)\sin nt \\ \phi^3 = \Sigma\, b_n(r)\sin nt\end{gathered} \tag{2.13}$$

converge in the mean to C^1 functions of r and t.

The equations for the Fourier coefficients,

$$a_{nrr} + \frac{2}{r}a_{nr} + n^2 a_n = b_n \tag{2.14}$$

$$b_n = 2\langle\sin nt\,(\Sigma\, a_k \sin kt)^3\rangle$$

have the formal solutions ($a_{nr}(0) = 0$),

$$
\begin{aligned}
a_n(r) &= \left(\alpha_n + \int_0^r \sigma b_n(\sigma) \cos n\sigma \, d\sigma\right) \frac{\sin nr}{nr} \\
&\quad - \left(\int_0^r \sigma b_n(\sigma) \sin n\sigma \, d\sigma\right) \frac{\cos nr}{nr} \qquad (2.15) \\
&= \frac{\kappa_n \sin (nr - \delta_n)}{nr} + \frac{1}{nr} \int_r^\infty \sigma b_n \sin n(\sigma - r) \, d\sigma
\end{aligned}
$$

where

$$\alpha_n \equiv a_n(0)$$

$$
\begin{aligned}
\kappa_n \cos \delta_n &= \alpha_n + \int_0^\infty r b_n \cos nr \, dr \\
\kappa_n \sin \delta_n &= \int_0^\infty r b_n \sin nr \, dr
\end{aligned}
\qquad (2.16)
$$

The asymptotic expansion of ϕ for $r \to \infty$,

$$\phi(r, t) = \sum \frac{\kappa_n \sin (nr - \delta_n)}{nr} \sin nt + \sum_2^\infty 0(r^{-n}) \qquad (2.17)$$

can be generated directly from (2.15). It follows that S, $\langle \phi_t^2 \rangle$ and $\langle \phi_r^2 \rangle$ are of $0(r^{-2})$.

3. *Approximate solutions*

Given the convergence of the Fourier series (2.13), it is suggested that qualitative features of increasingly more complicated solutions can be described by the Fourier-polynomials†,

$$
\begin{aligned}
\phi^{(\nu)} &= \sum_1^\nu a_n^{(\nu)}(r) \sin nt \\
(\phi^3)^{(\nu)} &= \sum_1^\nu b_n^{(\nu)}(r) \sin nt
\end{aligned}
\qquad (3.1)
$$

† Cf. Galerkin's method, the Rayleigh–Ritz method, the variational method.

The equations that govern the νth approximation are

$$a_n^{(\nu)}(r) = \left(\alpha_n^{(\nu)} + \int_0^r \sigma b_n^{(\nu)} \cos n\sigma\, d\sigma\right)\frac{\sin nr}{nr} - \left(\int_0^r \sigma b_n^{(\nu)} \sin n\sigma\, d\sigma\right)\frac{\cos nr}{nr} \tag{3.2}$$

$$b_n^{(\nu)}(r) = 2\left\langle \sin nt\left(\sum_1^{\nu} a_k^{(\nu)} \sin kt\right)^3\right\rangle$$

Note that $(\phi^3)^{(\nu)} \neq (\phi^{(\nu)})^3$; the terms where $n > \nu$ are discarded for both ϕ and ϕ^3.

The simplest of the Fourier-polynomials is the one-term approximation,

$$\phi \approx A(r) \sin t, \qquad \phi^3 \approx \tfrac{3}{4}A^3 \sin t \tag{3.3}$$

governed by

$$A_{rr} + \frac{2}{r}A_r + (1 - \tfrac{3}{4}A^2)A = 0, \qquad A_r(0) = 0 \tag{3.4}$$

To understand the behavior of $A(r)$, it is instructive to introduce the analog of $S[\phi]$,

$$2S[\phi] \to S_1[A] \equiv \tfrac{1}{2}(A_r^2 + A^2) - \tfrac{3}{16}A^4$$
$$\frac{dS_1}{dr} = A_r(A_{rr} + (1 - \tfrac{3}{4}A^2)A) = -\frac{2}{r}A_r^2 \tag{3.5}$$

The analog of the basin of $\phi = 0$ where periodic solutions are trapped is the interval $|A(0)| < 2/\sqrt{3}$ for which $0 \leqslant S_1[A(0)] < \frac{1}{3}$. The qualitative behavior of $A(r)$ can be seen in terms of the traces $(A(r), A_r(r))$ in the phase-plane: for $|A(0)| < 2/\sqrt{3}$ the trace spirals inward and, as $r \to \infty$,

$$A(r) = \frac{\kappa \sin(r - \delta)}{r} + \frac{9}{32}\frac{\kappa^3 \cos(r - \delta)}{r^2} + 0\left(\frac{1}{r^3}\right) \tag{3.6}$$

where $\kappa(A(0))(= -\kappa(-A(0)))$ and $\delta(|A(0)|)$ are monotonic functions, not easily evaluated. For $|A(0)| > 2/\sqrt{3}$, $|A(r)|$ increases and diverges at a finite value of r.

The specification of features of $A(r; \kappa)$ is a numerical problem, in general; here we shall discuss the two limits, $A(0) \to 0$ $(\kappa \to 0)$ and $A(0) \to 2/\sqrt{3}$ $(\kappa \to \infty)$.

In the limit where $A(0) \to 0$ we have $A^2 \ll 1$ for all r (from the properties of $S_1[A]$) and (3.2) can be iterated to obtain

$$A(r) = A(0)\left(\frac{\sin r}{r} - \tfrac{3}{4}A(0)^2 \frac{\cos r}{r}\int_0^r \frac{\sin^4 \sigma}{\sigma^2}\,d\sigma\right)(1 + 0(A^2)) \qquad (3.7)$$

It follows directly that the limit $A(0) \to 0$ is the limit $\kappa \to 0$ and that

$$\delta(\kappa) = \tfrac{3}{4}\kappa^2 \int_0^\infty \frac{\sin^4 r}{r^2}\,dr + 0(\kappa^4)$$

$$= \frac{3\pi}{16}\kappa^2 + 0(\kappa^4), \qquad \kappa \to 0 \qquad (3.8)$$

In the limit where $A(0) \to 2/\sqrt{3}$ (from below) the behavior of $A(r)$ is a nonuniform approach to the plane wave (i.e. the *non*-localized solution where $A \equiv 2/\sqrt{3}, A_r \equiv 0$) for finite values of r, and to the asymptotic decay (3.6) as $r \to \infty$. In effect, the domain of validity of (3.6), $r > R_*$ (say), is pushed outward, and κ and δ become larger and larger. The asymptotic form of $A(r)$ consists of a *body* which can be defined as $0 \leqslant r \leqslant R$ where R is the radius of the first zero of A, a *transition* $(R < r < R_*)$ where the oscillation is of relatively large magnitude and the separation of zeros of A is of $0(\ln R)$, and a *tail* where (3.6) is an adequate approximation. R, R_*, κ and δ all diverge as $A(0) \to 2/\sqrt{3}$; to see *how*, we consider the limit $R \to \infty$.

In the limit where $R \to \infty$ the limiting form of the solution in the body is

$$A(r) = \frac{2}{\sqrt{3}} \tanh\left[\frac{1}{2}\ln\left(\frac{r \sinh \sqrt{2}R}{R \sinh \sqrt{2}r}\right) + 0\left(\frac{1}{R}\right)\right] \qquad (3.9)$$

on the interval $0 \leqslant r \leqslant R$. The correction is *zero* at $r = R$ and is $0(1/R)$ on the closed interval. Thus as $R \to \infty, A(0) \to 2/\sqrt{3}$ in accord with the asymptotic relation,

$$A(0) \sim \frac{2}{\sqrt{3}}(1 - 4\sqrt{2}\,R\,e^{-\sqrt{2}R}) \qquad (3.10)$$

The qualitative behavior of the solution in the transition from body to tail can be inferred from the approximation, analogous to (3.9),

$$A(r) = \frac{2}{\sqrt{3}} \tanh\left[\frac{1}{2}\ln\left(\frac{\cosh \sqrt{2}(\bar{R} - r)}{\cosh \sqrt{2}(\bar{R} - R)}\right) + 0\left(\frac{1}{R}\right)\right] \qquad (3.11)$$

for $R \leqslant r \leqslant 2\bar{R} - R$, where $\bar{R}$ is the zero of A_r after R and $2\bar{R} - R$ is (approximately) the second zero of A. To obtain estimates of $A(\bar{R})$ and $\bar{R}$

we note that $2/\sqrt{3} - A(0)$ and $\frac{1}{3} - S_1(0)$ are exponentially small and that $S_1(\bar{R})$ can be evaluated approximately as

$$S_1(\bar{R}) = S_1(0) - \frac{8}{3R} \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{R-r}{\sqrt{2}}\right) dr + 0\left(\frac{1}{R^2}\right) \tag{3.12}$$

$$= \frac{1}{3} - \frac{32\sqrt{2}}{9R} + 0\left(\frac{1}{R^2}\right)$$

Since $A_r(\bar{R}) = 0$, (3.12) implies

$$A(\bar{R}) \sim -\frac{2}{\sqrt{3}}\left(1 - \left(\frac{8\sqrt{2}}{3R}\right)^{1/2}\right) \tag{3.13}$$

and from (3.11) it follows that

$$\bar{R} - R \sim \frac{1}{2\sqrt{2}} \ln\left(\frac{6R}{\sqrt{2}}\right) \tag{3.14}$$

Thus, as $A(0) \to 2/\sqrt{3}$ and $R \to \infty$, the amplitude of the oscillation in the transition is large

$$|\Delta A| \sim \frac{4}{\sqrt{3}}(1 - 0(R^{-1/2}))$$

and the separation of zeros of A is $0(\ln R)$, as mentioned earlier.

Once again from the properties of $S_1[A]$ it follows that the scale-length defined by the rate of change of $S_1(r)(S_1'(r)/S_1 = 0(1/r))$ increases with r while the separation of zeros of A decreases with r. Thus the estimate, period/scale-length $= 0(R^{-1}\ln R)$, is an overestimation for $r > R$; and a relatively gross approximation of the solution in the tail (error of $0(R^{-1}\ln R)$) can be found by lumping the transition and tail together in a single phase-amplitude approximation. For $R \leqslant r < \infty$ we seek a solution of the form $A(r) = \tilde{A}(P(r), r)$ where $\tilde{A}(\theta, r)$ is a 2π-periodic function of θ. Then, in the lowest approximation where $\partial/\partial r$ is neglected in comparison with $P_r\, \partial/\partial\theta$, we obtain

$$A(r) = f(P(r); E(r))(1 + 0(R^{-1}\ln R)) \tag{3.15}$$

where the osculating approximation $f(\theta; E)$ satisfies the local energy equation,

$$\tfrac{1}{2}P_r^2 f_\theta^2 + \tfrac{1}{2}f^2 - \tfrac{3}{16}f^4 = E(r) \tag{3.16}$$

The functions $P(r)$ and $E(r)$ are determined by the conditions, (1) that f shall be a 2π-periodic function of θ and (2) that the correction to $\tilde{A}$ of $0(R^{-1}\ln R)$ shall be a 2π-periodic function of θ (i.e. that there is no unbounded secular term)(Luke, 1966). The results can be most easily stated in terms of the action,

$$J(P_r, E) \equiv P_r^2 \oint f_\theta \, df = P_r j(E)$$

$$j(E) \equiv \oint \sqrt{(2(E - \tfrac{1}{2}f^2 + \tfrac{3}{16}f^4))}\,|df| \tag{3.17}$$

Then

$$(1) \leftrightarrow J_E = P_r j'(E) = 2\pi \tag{3.18}$$

$$(2) \leftrightarrow (r^2 J_{P_r})_r = (r^2 j(E))_r = 0 \tag{3.19}$$

the first is the nonlinear dispersion relation and the second is the transport equation $-r^2 j(E)$ is the adiabatic invariant.

In principle, we have $\kappa(R)$ and $\delta(R)$ approximated to $0(R^{-1} \ln R)$ now. As $r \to \infty; E \to 0; j(E) = 2\pi E(1 + \frac{9}{32}E + 0(E^2))$ and $A \sim \sqrt{2E} \sin P$: the appropriate choice of the integration constant for (3.19) is

$$j(E) = \frac{\pi\kappa^2}{r^2}, \qquad A \sim \frac{\kappa \sin P}{r} \tag{3.20}$$

At $r = R; E(R) \sim S_1(R) \sim \frac{1}{3}$ and $j(\frac{1}{3}) = 16\sqrt{2}/9$: thus

$$\kappa = \left(\frac{16\sqrt{2}}{9\pi}\right)^{1/2} R \left(1 + 0\left(\frac{\ln R}{R}\right)\right) \tag{3.21}$$

The evaluation of $\delta(R)$ is more complicated: what is called for is

$$P(r) = \pi + \int_R^r \frac{2\pi\, d\sigma}{j'(E)} = r - \delta(R) + 0\left(\frac{1}{r}\right) \tag{3.22}$$

The exact evaluation of the leading term of $\delta(R)$ appears to be rather difficult–various approximations lead finally to the numerical estimate,

$$\delta(\kappa) \approx 1{\cdot}57\,\kappa \left(1 + 0\left(\frac{\ln \kappa}{\kappa}\right)\right), \qquad \kappa \to \infty \tag{3.23}$$

with an error estimated at 1 or 2%.

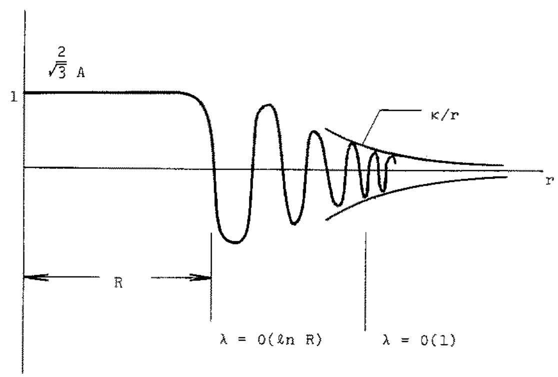

Figure 2. $R(\kappa) \to \infty$ and $\kappa \to \infty$.

In short, the functions $A(r; \kappa)$ vary smoothly from the linearized solution $A \sim (\kappa/r) \sin r$ for $\kappa \sim 0$ to the solution depicted schematically in Fig. 2 for $\kappa \to \infty$. The phase shift $\delta(\kappa)$ of the asymptotic approximation, $A \sim (\kappa/r) \sin(r - \delta)$ as $r \to \infty$, increases from $3\pi\kappa^2/16$ for $\kappa \sim 0$ to $\delta \approx \frac{1}{2}\pi\kappa$ as $\kappa \to \infty$.

4. *Heuristic theory of the discrete spectrum*

The one-term approximation

$$\phi(r, t) \approx A(r; \kappa) \sin t = \frac{\kappa}{r} \sin(r - \delta(\kappa)) \sin t + 0\left(\frac{1}{r^2}\right) \tag{4.1}$$

discussed in Section 3, is attractive in a number of ways: The approximate solutions of the form (4.1) describe the least complicated wavepackets that can be identified with free particles in the $(\phi^4)_4$-theory; and they provide a link between the wave-equation and the Klein–Gordon equation: both can be considered as governing asymptotic approximations ($\kappa \to 0$ and $\kappa \to \infty$, respectively) within the larger theory.

In this section we shall introduce the question of *when* (i.e. which values of κ) there is a periodic solution $\phi(r, t)$ that is 'close' to an approximation of the form (4.1). Evidently, something like a norm will be needed; we shall discuss the matter in terms of the total energy.

As noted in Section 1, the mean value of the energy within a sphere of radius R is constant when the solution is periodic. Accordingly, we shall define

$$\bar{E}_R \equiv 4\pi\left\langle \int_0^R r^2(\tfrac{1}{2}(\phi_t^2 + \phi_r^2) + \tfrac{1}{4}\phi^4)\, dr \right\rangle \tag{4.2}$$

where

$$\langle\cdot\rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} .\, dt \tag{4.3}$$

When $\phi(r, t)$ is a solution, $\bar{E}_R$ can be transformed by integrations by parts and the use of (1.3) and/or (1.4); e.g.

$$\begin{aligned} \bar{E}_R &= 4\pi\left\langle \tfrac{1}{2}r^2\phi\phi_r|_0^R + \int_0^R r^2(\phi_t^2 - \tfrac{1}{4}\phi^4)\, dr \right\rangle \\ &= 4\pi\left\langle \tfrac{1}{3}r^3\sigma|_0^R + \tfrac{2}{3}\int_0^R r^2(\phi_r^2 + \tfrac{3}{4}\phi^4)\, dr \right\rangle \end{aligned} \tag{4.4}$$

The most convenient form is the integrated form, i.e. the linear combination of the three that contains no integral over r,

$$\bar{E}_R = 4\pi R\langle \tfrac{1}{2}((r\phi)_t^2 + (r\phi)_r^2 - \phi^2) - \tfrac{1}{4}r^2\phi^4\rangle|_{r=R} \tag{4.5}$$

From (4.5) the limit of $\bar{E}_R$ as $R \to \infty$ can be evaluated in terms of $\{\kappa_n\}$. Let $r\phi = \chi$: then

$$\chi_{tt} - \chi_{rr} + \chi^3/r^2 = 0 \tag{4.6}$$

and, after one more integration by parts,

$$\begin{aligned} \bar{E}_R &= 4\pi R \left\langle \left(\frac{1}{2}(\chi_t^2 + \chi_r^2) - \frac{1}{4}\frac{\chi^4}{r^2}\right)\Bigg|_{r\to\infty} - \frac{1}{2}\int_R^\infty \frac{\chi^4}{r^3}\,dr - \frac{1}{2}\frac{\chi^2}{r^2}\Bigg|_{r=R} \right\rangle \\ &= \pi R\,\Sigma\kappa_n^2 + 0\left(\frac{1}{R}\right) \end{aligned} \tag{4.7}$$

The divergence of $\bar{E}_R$ with R, while in one sense less serious than that encountered in the quantized $(\phi^4)_4$-theory ($0(R)$ vs. $0(R^3)$), is in another sense more serious: renormalization by the disregarding of infinite contributions is not an option here—the renormalized energy of every periodic solution is *zero.* For present purposes we shall simply let $\bar{E}_R$ diverge, keeping track of how.

The coefficient $A(r)$ of the one-term approximation is governed by

$$A_{rr} + \frac{2}{r}A_r + (1 - \tfrac{3}{4}A^2)A = 0, \qquad A_r(0) = 0 \tag{4.8}$$

and the evaluation of $\bar{E}_R$ for $\phi = A(r)\sin t$ can be carried out in parallel with (4.4)-(4.7). The result,

$$\bar{E}_R = \pi R\kappa^2 + 0\left(\frac{1}{R}\right) \qquad (\phi = A(r;\kappa)\sin t) \tag{4.9}$$

suggests the first condition we shall propose as necessary if $\phi(r, t; \alpha_n)$ is to be well-represented by $A(r;\kappa)\sin t$; *viz.*

$$\kappa^2 = \Sigma\kappa_n^2. \tag{4.10}$$

To arrive at a relatively simple description of a spectrum of values of κ we shall consider the one-term approximation (4.1) as the first step of an iterative procedure that is to be used, in principle, to generate solutions of (1.3). At the third step and thereafter in such procedures there are relatively arbitrary decisions to be made with regard to the ordering of succeeding corrections; but the second step is the same in all cases, *viz.*

$$\phi(r, t) \approx A(r;\kappa)\sin t + \bar{A}(r;\kappa,\bar{\kappa})\sin 3t \tag{4.11}$$

where A is a solution of (4.8) and $\bar{A}$ is a solution of

$$\bar{A}_{rr} + \frac{2}{r}\bar{A}_r + 9\bar{A} = -\tfrac{1}{4}A^3, \qquad \bar{A}_r(0) = 0 \tag{4.12}$$

The solution of (4.12), given $A(r)$, can be written,

$$\bar{A}(r) = \frac{\bar{\kappa}\sin(3r - \bar{\delta})}{3r} + \frac{1}{12r}\int_r^\infty \sigma A^3(\sigma)\sin 3(r-\sigma)\,d\sigma \qquad (4.13)$$

where

$$\begin{aligned} \bar{\kappa}\cos\bar{\delta} &= \bar{A}(0) - \tfrac{1}{4}\int_0^\infty \sigma A^3 \cos 3\sigma\,d\sigma \\ \bar{\kappa}\sin\bar{\delta} &= -\tfrac{1}{4}\int_0^\infty \sigma A^3 \sin 3\sigma\,d\sigma \end{aligned} \qquad (4.14)$$

At this level of approximation the condition (4.10), that the energy of the one-term approximation shall be that of the solution generated from it, implies $\bar{\kappa} = 0$; i.e.

$$\bar{A}(0) = \tfrac{1}{4}\int_0^\infty \sigma A^3 \cos 3\sigma\,d\sigma \qquad (4.15)$$

$$\int_0^\infty \sigma A^3 \sin 3\sigma\,d\sigma = 0$$

The first of the conditions determines the integration constant as a function of κ; the second determines a discrete spectrum $\{\kappa_N\}$ of values of κ for which there exists a second approximation whose total energy does not differ from that of the first by an infinite amount of $0(R)$.

The evaluation of $\{\kappa_N\}$ is evidently a fairly involved numerical problem: here we shall present an estimate based on an approximation of $A(r)$ for large values of κ. In accord with the asymptotic form of $A(r;\kappa)$ (compare Section 3) we propose the rough approximation,

$$A(r;\kappa) \to \begin{cases} A_0 \approx 2/\sqrt{3} & 0 < r < R \\ \dfrac{\kappa\sin(r-\delta)}{r} & R < r < \infty \end{cases} \qquad (4.16)$$

where R is the first zero of A and κ is proportional to R. Then

$$\int_0^\infty \sigma A^3 \sin 3\sigma\,d\sigma \to A_0^3\int_0^R \sigma\sin 3\sigma\,d\sigma + \kappa^3\int_R^\infty \frac{\sin^3(\sigma-\delta)\sin 3\sigma\,d\sigma}{\sigma^2}$$

$$= 0(A_0^3 R) - \frac{\kappa^3}{8R}\cos 3\delta + 0(\kappa^3/R^2) \qquad (4.17)$$

and as $R \to \infty$, the term that has been evaluated is dominant. Thus the asymptotic estimate, for large κ, is (compare (3.23))

$$\delta_N \approx \frac{\pi}{3}(N + \tfrac{1}{2}), \qquad \kappa_N \approx \tfrac{2}{3}(N + \tfrac{1}{2}) \tag{4.18}$$

Just how accurate the estimates (4.18) are, especially for small N, will be left unspecified; at the very least, they can be regarded as a rough model of the actual spectrum.

The heuristic theory of the spectrum, though relatively easy, is not altogether satisfactory. In the first place, no compelling argument has been advanced in support of the hypothesis (4.10). Even if that is accepted on faith, however, further shortcomings appear when we consider the higher approximations. Let us suppose that the problem of specification of the order in which succeeding steps are to be carried out has been resolved; then the basic problem we are faced with can be stated as follows: At the νth step of the iteration we have k_ν Fourier coefficients, where k_ν depends on the scheme and generally increases with ν. Thus we have k_ν integration constants and only one relation between them. The discrete *non*degenerate spectrum of second approximations is fortuitous—the approximations of higher order are degenerate in a way that depends on the scheme.

In the section to follow we shall seek the further conditions that serve to resolve the degeneracy of the higher approximations. Then in Section 6 we shall develop a more comprehensive theory of the discrete spectrum.

5. *The variational problem*

The 'spectrum' of solutions of I-IV is nothing like discrete: the parameters that distinguish one solution from another (e.g. $\{\alpha_n\}$) define uncountably many solutions (aleph-1 to the power aleph-0). In this section we shall develop a criterion that reduces their number to a one-parameter family of countable sets (aleph-1 times aleph-0), still uncountable but considerably more manageable.

First, let it be noted that (1.1) is a necessary condition for the variational problem, $\delta W = 0$, whenever $W[\phi]$ exists. Accordingly, it is suggested that we regard that as the basic problem, with (1.1) a part of it. Then, in principle, all that need be done is to evaluate $W(\alpha_n) \equiv W[\phi(r, t; \alpha_n)]$ and investigate conditions for stationarity of W with regard to the choice of $\{\alpha_n\}$. For periodic solutions, the integration over all time can be replaced by the average over a period, and then, after the usual integrations by parts and the use of (1.3) and (2.17),

$$\bar{W}_R = 4\pi\left\langle \int_0^R r^2\left(\tfrac{1}{2}(\phi_t^2 - \phi_r^2) - \tfrac{1}{4}\phi^4\right) dr \right\rangle$$

$$= \pi\left(\left\langle \int_0^R r^2\phi^4\, dr \right\rangle - \sum \frac{\kappa_n^2}{2n} \sin 2(nR - \delta_n)\right) + 0\left(\frac{1}{R}\right) \tag{5.1}$$

Equation (5.1) displays a familiar annoyance that is known to appear in singular, two-point boundary value problems: when the interval becomes infinite, quantities we would like to manipulate either diverge, as we have seen with the energy, or cease to exist, as above. The difficulty is easily overcome, in this problem, by noting that $\bar{W}_R$ can be modified by the addition to it of a surface integral.

We assert that (1.2) was in fact the wrong guess, and that the appropriate variational principle for the spherically symmetric periodic solutions is $\delta\bar{W} = 0$ with $\bar{W}[\phi]$ *re*defined as the limit where $R \to \infty$ of

$$\bar{W}_R \equiv 4\pi \left\langle \int_0^R \{r^2(\tfrac{1}{2}(\phi_t^2 - \phi_r^2) - \tfrac{1}{4}\phi^4) + \tfrac{1}{2}(r^2\phi\phi_r)_r\}\, dr \right\rangle \tag{5.2}$$

The conditions for $\delta\bar{W}_R = 0$ are: (a) the Euler equation (1.3) on the interval $(0, R)$ and (b)

$$2\pi\langle r^2(\phi_r\,\delta\phi - \phi\delta\phi_r)\rangle|_0^R = 0 \tag{5.3}$$

Given a solution $\phi(r, t; \alpha_n)$, in accord with (a), the second condition for $\delta\bar{W} = 0$ can be written,

$$\lim_{r\to\infty} \langle r^2\Sigma\,(\phi_r\phi_{\alpha_n} - \phi\phi_{\alpha_n r})\,\delta\alpha_n\rangle = 0 \tag{5.4}$$

Moreover, the limit of $\bar{W}_R\,[\phi(r, t; \alpha_n)]$ exists and has the value,

$$\begin{aligned}\bar{V}(\alpha_n) &\equiv \lim_{R\to\infty} \bar{W}_R\,[\phi(r, t; \alpha_n)] \\ &= \pi\left\langle \int_0^\infty r^2\phi^4\, dr \right\rangle\end{aligned} \tag{5.5}$$

The partial derivatives of $\bar{V}(\alpha_n)$ exist and can be evaluated as follows: ϕ_{α_n} satisfies the variational equation,

$$\phi'_{tt} - \phi'_{rr} - \frac{2}{r}\,\phi'_r + 3\phi^2\phi' = 0 \tag{5.6}$$

and from (5.6) and (1.3) it follows that

$$\begin{aligned}\bar{V}_{\alpha_n} &\equiv 4\pi\left\langle \int_0^\infty r^2\phi^3\phi_{\alpha_n}\, dr \right\rangle \\ &= 2\pi \lim_{r\to\infty} \langle r^2(\phi\phi_{\alpha_n r} - \phi_r\phi_{\alpha_n})\rangle \\ &= \pi\sum \frac{\kappa_m^2}{m}\left(\frac{\partial\delta_m}{\partial\alpha_n}\right)\end{aligned} \tag{5.7}$$

By similar calculations,

$$\sum \frac{\kappa_m}{m} (\kappa_{m\alpha_i} \delta_{m\alpha_j} - \kappa_{m\alpha_j} \delta_{m\alpha_i}) = 0 \tag{5.8}$$

and

$$\bar{V}_{\alpha_i\alpha_j} = \pi \sum \frac{\kappa_m}{m} (\kappa_{m\alpha_i} \delta_{m\alpha_j} + \kappa_{m\alpha_j} \delta_{m\alpha_i} + \kappa_m \delta_{m\alpha_i\alpha_j}) \tag{5.9}$$

Equations (5.7) and (5.9) serve to relate properties of $\bar{V}(\alpha_n)$ and 'boundary conditions at ∞'; the condition (5.3) becomes

$$\delta \bar{V} \equiv \sum \bar{V}_{\alpha_n} \delta\alpha_n = 0 \tag{5.10}$$

What remains to be done is to decide what class of variations is to be allowed in the application of condition (5.10). The simplest variational problem, that $\bar{V}$ shall be minimized by the choice of $\{\alpha_n\}$, is too restrictive: $\bar{V}(\alpha_n)$ is greater than or equal to zero, and the unconstrained minimum, $\bar{V} = 0$, is clearly attained for $\alpha_n = 0$, i.e. for the trivial solution, $\phi = 0$. There might be other stationary points of $\bar{V}(\alpha_n)$,† but, if such points exist, the corresponding solutions are not generated by the one-term approximation (4.1). In the one-term approximation $\bar{V}(\alpha_n) \to \bar{V}(\alpha)$, $\alpha \equiv A(0)$, and (from the results of Section 3) the only zero of $\bar{V}_\alpha$ is at $\alpha = 0$.

To arrive at a variational problem that includes nontrivial solutions, we propose that $\bar{V}$ shall be minimized subject to a constraint, $\bar{H}(\alpha_n) =$ constant, Now let it be noted that there is a 'constant of the motion' associated with the problem, $\delta \bar{W}_R = 0$, *viz* the total energy,

$$\bar{E}_R = \bar{T}_R + \bar{V}_R \tag{5.11}$$

$$\bar{T}_R \equiv 2\pi \left\langle \int_0^R r^2 (\phi_t^{\,2} + \phi_r^{\,2})\, dr \right\rangle, \qquad \bar{V}_R \equiv \pi \left\langle \int_0^R r^2 \phi^4\, dr \right\rangle$$

The finite problems, that $\bar{W}_R$ shall be minimized subject to the constraint, $\bar{E}_R =$ constant, (essentially the principle of least action), have a well defined limit with the necessary conditions, (1.3) on the interval $0 < r < \infty$ and

$$\delta \left(\frac{\bar{V}}{\bar{H}}\right) = 0, \qquad \bar{H} \equiv \lim_{R\to\infty} \frac{1}{R} \bar{T}_R = \pi \sum \kappa_n^{\,2} \tag{5.12}$$

The corresponding conditions,

$$\sum \kappa_m \left(\frac{\kappa_m}{m} \delta_{m\alpha_n} - 2\lambda \kappa_{m\alpha_n}\right) = 0, \qquad \lambda = \frac{\bar{V}}{\bar{H}} \tag{5.13}$$

† This would not be an option if $\bar{V}(\alpha_n)$ were convex; a proof of that, if true, would strengthen the arguments of this section.

are the heretofore unspecified 'boundary conditions at ∞' (complemented by $a_{nr}(0) = 0$) that provide the means for dealing with the degeneracy of $\phi(r, t; \alpha_n)$. The global structure of the resulting problem can be conjectured to be the following:

We shall *assume* that $\bar{V}(\alpha_n)$ varies on the surface, $\bar{H}(\alpha_n)$ = constant, in such manner that (5.13) has a discrete spectrum of solutions, $\{\alpha_n^{(N)}(\bar{H})\}$.† In general, the functions $\alpha_n^{(N)}(\bar{H})$ cannot be expected to be continuous, much less differentiable: to 'compute' them, even approximately, requires a change of the point of view. The problem can be approached by considering C^1-curves $\{\alpha_n(s)\}$ on which $d\bar{H}/ds = \Sigma\dot{\alpha}_n\bar{H}_{\alpha_n} \neq 0$. Then, if $\{\alpha_n^{(N)}(\bar{H})\}$ is discontinuous, the problem, find stationary points of $\bar{V}$ on $\bar{H}$ that are *near* a specific curve $\{\alpha_n(s)\}$, determines a discrete spectrum $\{S_N[\alpha_n(s)]\}$ and $\{\bar{H}(\alpha_n(S_N))\}$. To find all the stationary points in a relatively systematic manner one can consider: first the α_1-axis, then the rays in the α_1, α_2-plane, and so on.

The foregoing conjectured geometrical interpretation of the appearance of discrete spectra in the classical $(\phi^4)_4$-theory is partially supported by the result of the section to follow. On the α_1-axis and with *near* interpreted as sufficiently near to be found by successive approximations, (5.13) determines the spectrum $\{\kappa_N\}$ of Section 4—without (4.10) and nondegenerate.

6. *Bifurcation–formal theory of the discrete spectrum*

The problem at hand can be characterized as one of *singular bifurcation theory: singular* because of the infinite interval and related unmanageable degeneracy of the solutions $\phi(r, t; \alpha_n)$—solutions we shall view as branching from the one-term approximation at a spectrum of *bifurcation points* ($\{\kappa_N\}$ or $\{\delta_N\}$) (Keller & Antman, 1969). In some problems of *regular degenerate bifurcation theory* it has been possible to solve the bifurcation problem (with fixed finite degeneracy), then resolve the degeneracy by an appeal to *stability* (Dean & Chambre, 1970). That, however, is not possible here; and we proposed to reverse the order, dealing with stability first, bifurcation after.

The stability problem associated with solutions of (I-IV) is in fact a rather involved one: it is necessary to consider perturbed solutions of (1.1) within a relatively broad class of *non*periodic functions. For present purposes we shall avoid the stability problem by presenting, as a conjecture, that (5.13) is a stability condition.‡ What will be done here is to develop a *formal* scheme of successive approximations for (1.3) *and* (5.13), based on the intuitive notion that we are dealing with solutions for which $\phi \approx A \sin t$. The condition (5.13) provides an eigenvalue problem that determines successive approximations of $\lambda(=\bar{V}/\bar{H})$ and $\{\alpha_n\}$: the bifurcation condition will be estimated from the first approximate solution of the eigenvalue problem.

† Here again a proof would be welcome, and would strengthen the argument about global structure; local structure (where $\phi \approx A \sin t$) will be dealt with, shortly.

‡ Stability will be discussed in a sequel.

Let us begin by isolating the first Fourier coefficients of functions $\phi(r, t)$ that satisfy condition II, in terms of projection operators. Let

$$p\phi \equiv \frac{1}{\pi}\int_0^{2\pi} \phi(r, t) \sin t \, dt$$

$$P\phi \equiv \sin t \, p\phi$$

$$\bar{P}\phi \equiv \phi - P\phi \tag{6.1}$$

and in particular, for a solution,

$$\phi = P\phi + \bar{P}\phi = a(r) \sin t + \bar{\phi}, \qquad p\bar{\phi} = 0 \tag{6.2}$$

The basic conjecture, at the first level of description, is that solutions of (1.3) in a neighborhood of a solution of (5.13) can be generated by the iteration,

$$a(r; \alpha_n)\colon a(0; \alpha_n) = \alpha_1 \tag{6.3}$$

$$a_{rr} + \frac{2}{r} a_r + (1 - \tfrac{3}{4}a^2)a = p(\phi^3 - (P\phi)^3)$$

$$\bar{\phi}(r, t; \alpha_n)\colon \bar{\phi}(0, t; \alpha_n) = \sum_{n \neq 1} \alpha_n \sin nt \tag{6.4}$$

$$\bar{\phi}_{rr} + \frac{2}{r}\bar{\phi}_r - \bar{\phi}_{tt} = -\tfrac{1}{4}a^3 \sin 3t + \bar{P}(\phi^3 - (P\phi)^3)$$

Indices have been suppressed in favor of the convention that the *declared* quantity, a or $\bar{\phi}$, is the only unknown in the problem that follows it. In principle, the cyclic iteration ((6.3), (6.4), (6.2)), with the initial condition $\bar{\phi} = 0$, is to be performed for sufficiently many cycles and for sufficiently many values of $\{\alpha_n\}$ to obtain an approximate solution, with specified accuracy, of (5.13).

As posed at this level, the iteration is

$$\bar{\phi} = 0, ((6.3), (6.4), (6.2))^\nu, (5.13) \tag{6.5}$$

where the exponent ν indicates the number of times the cycle is iterated. Since (5.13) is outside the cycle, $\lambda(\alpha_n)$ is to be determined for arbitrary $\{\alpha_n\}$ (in a neighborhood of a solution of (5.13)). Needless to say, an iteration that contains (5.13) in the cycle is a great deal more efficient.

There are many ways to modify (6.5) so that (1.3) and (5.13) are dealt with simultaneously: they can be written symbolically as

$$\bar{\phi} = 0, (6.3), (6.4), (6.2), (5.13), (X_1), \ldots, (X_\nu) \tag{6.6}$$

where the steps (X_k) evaluate corrections (often by Newton's method) for both (1.3) and (5.13). A specific example of an iteration of the form (6.6) can be exhibited by evaluating, at $\epsilon = 1$, the perturbation theory that follows when $\phi^3 - (P\phi)^3 \to \epsilon(\phi^3 - (P\phi)^3)$ in (6.3, 6.4), $\phi \to \Sigma\, \epsilon^n \phi_n$ and $\{\alpha_k\} \to \{\Sigma\, \epsilon^n \alpha_{kn}\}$.

Generally speaking, the step (X_k) can be characterized as follows: given $\{\alpha_n\}$ from (X_{k-1}) one corrects ϕ (and possibly ϕ_{α_n} and $\phi_{\alpha_i\alpha_j}$) in accord with (1.3) and re-evaluates $\lambda(\alpha_n)$, λ_{α_n} and $\lambda_{\alpha_i\alpha_j}$. Since ϕ has been modified, λ_{α_n} is not zero at $\{\alpha_n\}$, except by accident; the simplest estimate of the corrected values $\{\alpha_n\}$ is (Newton's method)

$$\Sigma \lambda_{\alpha_i\alpha_j} \delta\alpha_j + \lambda_{\alpha_i} = 0 \tag{6.7}$$

$$\alpha_n \leftarrow \alpha_n + \delta\alpha_n$$

Equation (6.7) exhibits, in a fairly general way, the importance of a *solvability* condition. The infinite determinant $|\lambda_{\alpha_i\alpha_j}|$ is not well-defined; the equivalent condition,

$$|\Delta_{ij}| \neq 0, \qquad \Delta_{ij} \equiv \frac{1}{2\pi}\left(\bar{H}_{\alpha_i\alpha_j} - \frac{1}{\lambda}\bar{V}_{\alpha_i\alpha_j}\right) \tag{6.8}$$

is necessary and sufficient for unique determination of $\{\delta\alpha_n\}$.

The leading approximation of (6.6) (or (6.5)) is

$$(X_0) = (\bar{\phi} = 0, (6.3), (6.4), (6.2), (5.13)) \tag{6.9}$$

and, except for (5.13), the general solution is

$$\phi(r, t; \alpha_1, \alpha_3, \{\alpha_n\}') = A_1(r) \sin t + A_3(r) \sin 3t + \Sigma' a_n(r) \sin nt \tag{6.10}$$

where $\{\alpha_n\}'$ and Σ' exclude $n = 1$ and 3 and

$$A_1(r) = A(r; \kappa(\alpha_1))$$

$$A_3(r) = \left(\alpha_3 - \tfrac{1}{4}\int_0^r \sigma A^3 \cos 3\sigma\, d\sigma\right)\frac{\sin 3r}{3r} + \tfrac{1}{4}\left(\int_0^r \sigma A^3 \sin 3\sigma\, d\sigma\right)\frac{\cos 3r}{3r} \tag{6.11}$$

$$a_n(r) = \alpha_n \frac{\sin nr}{nr}$$

There is a notable difference between the notions of *successive approximation* here and in Section 4: there, a number of *new* harmonics are introduced at each stage (compare (4.11)); whereas here, the harmonics are all present (including those where n is even) and it is new interactions that are successively introduced. The condition (5.13) determines the values of all the integration constants $\{\alpha_n\}$ at each stage, with the satisfying result that those in $\{\alpha_n\}'$ are zero.

The quantities that appear in the leading approximation of (5.13) are:

$$\kappa_1 = \kappa(\alpha_1) \qquad \text{and} \qquad \delta_1 = \delta(\kappa(\alpha_1))$$

as discussed in Section 3;

$$\kappa_3(\alpha_1, \alpha_3) = \bar{\kappa}(\bar{A}(0), \kappa) \qquad \text{and} \qquad \delta_3(\alpha_1, \alpha_3) = \bar{\delta}(\bar{A}(0), \kappa)$$

as defined by (4.14) with $\alpha_3 = \bar{A}(0)$; and, for $n \neq 1$ or 3,

$$\kappa_n = \alpha_n \qquad \text{and} \qquad \delta_n = 0$$

It follows directly from (5.13) ($n \neq 1$ or 3) that $\lambda = 0$ (*not* allowed) or $\{\alpha_n\}' = \{0\}'$, thus indicating the equivalence (in practice—not in principle) of the successive approximations here are in Section 4. The result, $\alpha_n = 0$ for a 'non-interacting' harmonic, is obtained at all stages of (6.6).

The two remaining conditions are

$$\kappa_1(\kappa_1\delta_{1\alpha_1} - 2\lambda\kappa_{1\alpha_1}) + \kappa_3(\tfrac{1}{3}\kappa_3\delta_{3\alpha_1} - 2\lambda\kappa_{3\alpha_1}) = 0 \tag{6.12}$$

$$\kappa_3(\tfrac{1}{3}\kappa_3\delta_{3\alpha_3} - 2\lambda\kappa_{3\alpha_3}) = 0$$

In the leading approximation the bifurcation condition has the two roots, $\lambda(\alpha_1, \alpha_3)$: $\kappa_1\delta_{1\alpha_1} = 2\lambda\kappa_{1\alpha_1}$ *or* $\kappa_3\delta_{3\alpha_3} = 6\lambda\kappa_{3\alpha_3}$; only the first is related to the problem at hand where $\phi \approx A \sin t$. Thus

$$\frac{\bar{V}}{\bar{H}} \equiv \lambda \approx \lambda(\alpha_1) = \tfrac{1}{2}\kappa\delta'(\kappa), \qquad \kappa(\alpha_1) = \kappa_1 \tag{6.13}$$

($\kappa\delta'(\kappa)$ varies smoothly from $3\pi\kappa^2/8$ ($\kappa \to 0$) to $\pi\kappa/2$, approximately ($\kappa \to \infty$).) It follows from (6.12) and (6.13) that κ is in the spectrum $\{\kappa_N\}$:

$$\kappa_3 = 0 \leftrightarrow \kappa = \kappa_N \approx \tfrac{2}{3}(N + \tfrac{1}{2}), \qquad \delta = \delta_N \approx \frac{\pi}{3}(N + \tfrac{1}{2}) \tag{6.14}$$

(compare Section 4).

The determinant $|\Delta_{ij}|$ is relatively easy to evaluate in the leading approximation. Equations (6.12), complemented by $-2\lambda\alpha_n = 0$ for $n \neq 1$ or 3, are the equations $(1/\pi)(\bar{V}_{\alpha_n} - \lambda\bar{H}_{\alpha_n}) = 0$; the α_j-derivative of the left-hand side of the ith condition is $-2\lambda\Delta_{ij}$. Since $\lambda_{\alpha_n} = 0$, λ acts like a constant in the calculation of Δ_{ij}; and the only entries that are not *ones* on the principal diagonal are those from (6.12). The further relations, $\kappa_3 = \kappa_{1\alpha_3} = \delta_{1\alpha_3} = 0$, $\kappa_1\delta_{1\alpha_1} = 2\lambda\kappa_{1\alpha_1}$ and $\kappa_{3\alpha_3} = 1$ (compare (4.14) which also implies $\sin\delta_3 = 0$), can be used to obtain

$$|\Delta_{ij}| = \kappa_1\left(\kappa_{1\alpha_1} - \frac{\kappa_1\delta_{1\alpha_1}}{2\lambda}\right)_{\alpha_1}$$

$$= -(\kappa_{\alpha_1}^2(\kappa\delta'(\kappa))'/\delta'(\kappa))\,|_{\kappa=\kappa_1=\kappa(\alpha_1)} \tag{6.15}$$

From the results of Section 3, the leading estimate of $|\Delta_{ij}|$ cannot vanish—when (6.6) converges the solution is nondegenerate, and the conjectures at the end of Section 5 are verified near the α_1-axis.

7. *The irresistible force–asymptotics*

The discrete spectrum of spherically symmetric, periodic solutions is based on three major conjectures:

(1) That it makes sense to delimit the class of regular, periodic solutions of (1.3) by the conditions of the associated variational problem. Some such conditions are necessary if we are to find a nondegenerate spectrum; the choice, a variational formulation, is known to be related to stability; but the stability theory has not been presented here.

(2) That there is a convergent iteration (6.6). There appears to be enough information here to suggest the *a priori* estimates necessary for a constructive existence proof; but of course one has not been provided.

(3) That the simplest solutions (near the α_1-axis) are somehow preferred. This is almost as arbitrary as *quantizing* the theory, but it can be *checked.* The simple solutions either *are* attractor in some wider class of problems or they are *not.*

In this section we shall accept the major conjectures and add quite a few others.

The basic problem here is to isolate that feature of a $(\phi^4)_4$-wavepacket that plays the role of mass of a $(\phi^4)_4$-particle. To be certain we have made the correct identification, it will be necessary to relax conditions I–IV and discuss accelerated wavepackets: a *very* sketchy theory of interactions will be presented.

We propose first to consider solutions of (1.1) that have the property,

$$\begin{aligned} \phi(\mathrm{x}, t) &= \phi_1(\mathrm{x}, t) + \phi_2(\mathrm{x}, t) \\ \rho_1 \ll \rho_2 &\qquad \text{in } R_2(t) \\ \rho_2 \ll \rho_1 &\qquad \text{in } R_1(t) \end{aligned} \tag{7.1}$$

where $R_1(t)$ and $R_2(t)$ are disjoint regions of space. Solutions of the form (7.1) can be regarded as interacting wavepackets, sufficiently well separated to preclude strong interactions and the creation and destruction of particles.

For such solutions we propose the provisional scheme of successive approximations,

$$\begin{aligned} &\phi_1(\mathrm{x}, t)\colon \phi_{1tt} - \Delta\phi_1 + \phi_1^3 + (\tfrac{3}{2} + \gamma_1)\phi_2\phi_1^2 + (\tfrac{3}{2} - \gamma_2)\phi_2^2\phi_1 = 0 \\ &\phi_2(\mathrm{x}, t)\colon \phi_{2tt} - \Delta\phi_2 + \phi_2^3 + (\tfrac{3}{2} + \gamma_2)\phi_1\phi_2^2 + (\tfrac{3}{2} - \gamma_1)\phi_1^2\phi_2 = 0 \end{aligned} \tag{7.2}$$

with the initial condition $\phi_{2tt} - \Delta\phi_2 + \phi_2^3 = 0$. The most that can be said, at this point, about the functions $\gamma_1(\mathrm{x}, t)$ and $\gamma_2(\mathrm{x}, t)$ is *that they cannot be arbitrary*: an arbitrary choice of (γ_1, γ_2) leads to an arbitrary interaction of two wavepackets; the Euler equation (1.1) determines a specific interaction. It appears that the γ's are to be determined by necessary conditions for the convergence of (7.2) to a solution of (1.1); and it can be conjectured that they are related to isospin and hypercharge of interacting scalar mesons.

We shall postpone the problem of determination of the γ's, and examine the generic problem in the iteration, *viz.*

$$\phi_{tt} - \Delta\phi + \phi^3 + W(\mathbf{x}, t)\phi^2 + V(\mathbf{x}, t)\phi = 0 \tag{7.3}$$

where $V(\mathbf{x}, t)$ and $W(\mathbf{x}, t)$ are specified potentials. The corresponding Lagrangian density is

$$\mathscr{L} \equiv \tfrac{1}{2}\phi_t^2 - \tfrac{1}{2}|\nabla\phi|^2 - \tfrac{1}{4}\phi^4 - \tfrac{1}{3}\phi^3 W - \tfrac{1}{2}\phi^2 V \tag{7.4}$$

and the conservation laws are

$$\begin{aligned} \rho_t + (\rho u_i)_{,i} &= \tfrac{1}{2}\phi^2 V_t + \tfrac{1}{3}\phi^3 W_t \\ (\rho u_i)_t + \pi_{ij,j} &= -\tfrac{1}{2}\phi^2 V_{,i} - \tfrac{1}{3}\phi^3 W_{,i} \end{aligned} \tag{7.5}$$

where

$$\begin{aligned} \rho &\equiv \phi_t \mathscr{L}_{\phi_t} - \mathscr{L} \\ &= \tfrac{1}{2}(\phi_t^2 + |\nabla\phi|^2) + \tfrac{1}{4}\phi^4 + \tfrac{1}{3}\phi^3 W + \tfrac{1}{2}\phi^2 V \\ \rho u_i &\equiv \phi_t \mathscr{L}_{\phi_{,i}} = -\phi_t \phi_{,i} \\ \pi_{ij} &\equiv \mathscr{L}\delta_{ij} - \phi_{,i}\mathscr{L}_{\phi_{,j}} = \mathscr{L}\delta_{ij} + \phi_{,i}\phi_{,j} \end{aligned} \tag{7.6}$$

There are quantum mechanical approximations of (7.3) (Bisshopp, 1971b); we shall bypasss them here, and isolate the classical particle mechanics. The basic hypothesis is that we are dealing with a solution that has a characteristic frequency that is *large*, i.e. that there is a local oscillation with a period that is negligibly small, compared with any other time-scale in the problem. Formally, we introduce a characteristic frequency $\omega(t)$ and consider the limit $\omega \to \infty$. The leading term of the asymptotic expansion of a solution of (7.3) is then

$$\phi(\mathbf{x}, t) = \omega(t)\tilde{\phi}(\boldsymbol{\xi}, \tau, \mathbf{x}, t)(1 + 0(\omega^{-1})) \tag{7.7}$$

where

$$\boldsymbol{\xi} = \omega(t)(\mathbf{x} - \mathbf{X}(t))(1 + 0(\omega^{-1})) \tag{7.8}$$

$$\tau = \int^t \omega(s)\, ds\, (1 + 0(\omega^{-1}))$$

and

$$\tilde{\phi}_{\tau\tau} - \Delta_\xi \tilde{\phi} + \tilde{\phi}^3 = 0 \tag{7.9}$$

To complete the description of the leading approximation for a classical particle we shall treat the special case (in accord with the major conjectures) where $\tilde{\phi}$ describes the Nth moving, spheroidal wavepacket (compare (1.12)); i.e.

$$\phi(\mathbf{x}, t) \sim \omega(t)\phi_N(\bar{r}, \bar{\tau}) \tag{7.10}$$

where

$$\bar{\tau} = \frac{\tau - \mathbf{U}.\boldsymbol{\xi}}{\sqrt{1 - U^2}}, \qquad \mathbf{U}(t) = d\mathbf{X}/dt \tag{7.11}$$

$$\bar{r}^2 = |\boldsymbol{\xi}|^2 - (\hat{U}.\boldsymbol{\xi})^2 + \frac{(\hat{U}.\boldsymbol{\xi} - U\tau)^2}{1 - U^2}, \qquad \hat{U} = \mathbf{U}/U$$

The equations of motion (for $\omega(t)$ and $\mathbf{X}(t)$) are now to be found by substitution of (7.10) and (7.11) in the conservation laws (7.5); the calculation is easiest in the nonrelativistic case where $U^2 \ll 1$. In the double limit, $\omega \to \infty$ and $U \to 0$, we have

$$\phi(\mathbf{x}, t) \sim \omega\phi_N(r, \tau)$$

$$r \sim |\boldsymbol{\xi}| = \omega|\mathbf{x} - \mathbf{X}(t)|$$

$$\tau \sim \int^t \omega\, ds$$

$$\phi_t \sim \omega^2\left(\phi_{N\tau} - \mathbf{U}.\boldsymbol{\xi}\frac{\phi_{Nr}}{r}\right) + \dot{\omega}(\phi_N + r\phi_{Nr}) \tag{7.12}$$

$$\nabla\phi \sim \omega^2\boldsymbol{\xi}\frac{\phi_{Nr}}{r}$$

$$d^3x \sim \frac{1}{\omega^3}d^3\xi = \frac{4\pi}{\omega^3}r^2\, dr$$

The next step is to compute ρ, ρu_i and π_{ij}, and then to evaluate the conservation laws, *averaged* (Whitham, 1965) over the rapid variation of ϕ with τ and then integrated over all space. Let $\langle\cdot\rangle$ denote the average over the period 2π of the τ-variation; then

$$\begin{aligned}\langle\rho\rangle &= \omega^4\langle\tfrac{1}{2}(\phi_{N\tau}^2 + \phi_{Nr}^2) + \tfrac{1}{4}\phi_N{}^4\rangle \\ &\quad + \omega^2\langle\tfrac{1}{2}\phi_N{}^2\rangle V + 0(\omega^4U^2) + 0(\omega^2\dot{\omega}) \\ \langle\rho u_i\rangle &= \omega^4\langle\phi_{Nr}^2\rangle U_i + 0(\omega^2\dot{\omega})\end{aligned} \tag{7.13}$$

Now $\langle\rho_t\rangle = \langle\rho\rangle_t$ and $\langle(\rho u_i)_{,i}\rangle \sim \omega^4 U_i\langle\phi_{Nr}^2\rangle_{,i}$; neglecting terms of higher order, we obtain for the first of equations (7.5), evaluated at $U = 0$,

$$\langle\rho\rangle_t \sim \langle\tfrac{1}{2}\phi^2\rangle V_t \tag{7.14}$$

The integral of (7.14) over all space is

$$\frac{\partial}{\partial t}\left(\lim_{R\to\infty}\left(\omega\bar{H}_R + \frac{1}{\omega}\bar{K}_R\right)\right) \sim \lim_{R\to\infty}\frac{1}{\omega}\frac{\partial\bar{K}_R}{\partial t} \tag{7.15}$$

where

$$\bar{H}_R \equiv 4\pi \int_0^R r^2 \langle \tfrac{1}{2}(\phi_{N\tau}^2 + \phi_{Nr}^2) \rangle \, dr \sim R\bar{H}_N$$

$$\bar{K}_R \equiv 4\pi \int_0^R r^2 \langle \tfrac{1}{2}\phi_N{}^2 \rangle V \, dr \sim R\bar{G}_N V(\mathbf{X}, t)$$

$$\bar{H}_N = \pi(\Sigma \kappa_n{}^2)_N, \qquad \bar{G}_N = \frac{\pi}{2}\left(\Sigma \frac{\kappa_n{}^2}{n^2}\right)_N \tag{7.16}$$

The two sides of (7.15) are compatible, and it follows directly that $\dot{\omega} = 0$—the rest frequency of a $(\phi^4)_4$ classical particle is *constant.*

It follows easily that the surface integrals of $\langle \pi_{ij} n_j \rangle$ are $0(1)$ as $R \to \infty$; the integrals of $r^2 \langle \phi_{Nr}^2 \rangle$ and $\frac{1}{2} r^2 \langle \phi_{N\tau}^2 + \phi_{Nr}^2 \rangle$ diverge at the same rate $(R\bar{H}_N/4\pi)$; and R is dimensionless. The remaining conservation laws, averaged twice and evaluated at $U = 0$, are

$$\frac{d}{dt}\left(\mu_0 \frac{dX_i}{dt}\right) \sim -\mu_1 \frac{\partial V}{\partial X_i} \tag{7.17}$$

where

$$\begin{aligned} \mu_0 &= \hbar\omega, & \hbar &= \bar{H}_N \\ \mu_1 &= g/\omega, & g &= \bar{G}_N \end{aligned} \tag{7.18}$$

The acceleration of the guiding center is $0(\omega^{-2})$ in the double limit $\omega \to \infty$, $U \to 0$; i.e. the nonrelativistic classical mechanics is governed by

$$\frac{d^2 X_i}{dt^2} = -\frac{g}{\hbar\omega^2} \frac{\partial V}{\partial x_i}\bigg|_{\mathbf{X}(t)}, \qquad \frac{g}{\hbar} = \frac{\bar{G}_N}{\bar{H}_N} \approx \tfrac{1}{2} \tag{7.19}$$

To find the relativistic mechanics, we note that (7.3) is invariant under the Lorentz transformation to the instantaneous rest-frame. The relativistic motion of the guiding center is determined implicitly by $(t, \mathbf{X}) = (X^\alpha(s))$ and $(t, -\mathbf{X}) = (X_\alpha(s))$:

$$\begin{aligned} \dot{X}^\alpha \dot{X}_\alpha &= 1 \\ (\mu \dot{X}_\alpha)^{\cdot} &= \mu_1 \frac{\partial V}{\partial X^\alpha} \end{aligned} \tag{7.20}$$

where

$$\mu(s) = \mu_0 + \mu_1 V(X^\alpha(s)) \tag{7.21}$$

The final result follows from $\dot{X}^\alpha (\mu \dot{X}_\alpha)^{\cdot} = \dot{\mu} = \mu_1 \dot{V}$ and the requirements that $(7.20) \to (7.19)$ in the double limit, $\omega \to \infty$ and $U \to 0$, and in the limit $V \to 0$. It may be noted that the effective rest mass is the sum of a constant bare mass and a correction that depends on the interaction, even in the classical limit.

Interactions according to (7.20) will be discussed, along with stability and stronger interactions, in a sequel; for the moment, let us simply observe that the interactions are *weak* in two senses: In the first place; given V, the interaction is vanishingly small when $\omega \to \infty$, as we have seen. Secondly; the potential that describes the interaction of one particle with another is $0(|\mathbf{r} - \mathbf{r}'|^{-2})$ in the non-relativistic case; i.e. the force-law is inverse cube, rather than inverse square. Moreover, $(\phi^4)_4$ lacks charge and spin, but quite possibly has hypercharge and isospin; it seems safe to assert that it cannot possibly describe particles more complicated than the scalar mesons, if indeed it does that.

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